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Irreversible Quantum Mechanics in the Neutral *K*-System

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The neutral kaon system is used to test the quantum theory of resonance scattering and decay phenomena. The two dimensional Lee–Oehme–Yang theory with complex Hamiltonian is obtained by truncating the complex basis vector expansion of the exact theory in rigged Hilbert space. This can be done for K_1 and K_2 as well as for K_S and K_L, depending upon whether one chooses the (self-adjoint, semibounded) Hamiltonian as commuting or noncommuting with *CP*. As an unexpected curiosity, one can show that the exact theory (without truncation) predicts long-time 2π decays of the neutral kaon system even if the Hamiltonian conserves *CP*.

1. THE STANDARD TWO-DIMENSIONAL EFFECTIVE THEORY WITH COMPLEX HAMILTONIAN

The phenomenological characteristics of a resonance or a decaying particle are its energy E_R (resonance energy) or relativistically its mass m, and its width $\Gamma = \hbar/\tau_R$, where τ_R is its lifetime (in its rest frame). These two real numbers are combined into a complex energy $z_R = E_R - i\,\gamma/2$ [or relativistically into a complex value $s_R = (m - i\,\gamma/2)^2$ of the Mandelstam variable s, which is the (energy)^2 in the rest frame of the decaying state or the (total energy)^2 in the center-of-mass frame of the decay products (often not m, but M defined by $s_R = M^2 - iM\gamma$ is called the mass of the resonance)]. This complex energy z_R or complex mass-squared s_R can be defined as the position of the first-order (resonance) pole in the second sheet of the analytically continued S-matrix, in the same way as stationary states are given by bound-state poles of the S-matrix. Empirically, stability or the value of the lifetime does not appear to be a criterion for elementarity. Stable particles are

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not qualitatively different from quasistable particles, but only quantitatively different by a zero or very small value of Γ. (A particle decays if it can decay and it is stable if selection rules for some quantum numbers prevent it from decaying.) Therefore both stable and quasistable states should be described on the same footing, as is the case if one defines them by S-matrix poles.

Since a stationary state characterized by a real energy value E_s is described not only by a bound-state pole on the negative real axis, but also by an eigenvector of the (self-adjoint, semibounded) Hamiltonian H with eigenvalue E_s, a "pure" decaying state should also be described by an "eigenvector" of H, but with complex eigenvalue z_R. In the standard, Hilbert space formulation of quantum mechanics such vectors do not exist. But since it is of practical importance for the phenomenological description of experiments to have a vector space description, this deficiency of the Hilbert space quantum mechanics has not prevented the practitioners from using eigenvectors with complex energy in "phenomenological", "effective" theories of decay (Gamow, 1928; Peierls, 1955; Garcia-Calderon and Peierls, 1976; Hernandez and Mondragon, 1984). The Lee–Oehme–Yang theory for the time evolution of the two-resonance neutral kaon system is the most celebrated example (Lee, 1981) of this.

The Lee–Ohme–Yang theory is usually justified as the Weisskopf–Wigner approximation applied to the two kaon states $|K^0\rangle$, $|\bar{K}^0\rangle$, in which the neutral K^0-system is prepared in an inelastic scattering experiment by strong interaction (e.g., $\pi^- p \to \Lambda K^0$).

One then introduces the two linear combinations $|K_1\rangle$ and $|K_2\rangle$ (leaving aside the CP-violating Hamiltonian)—or the $|K_S\rangle$ and $|K_L\rangle$ (if one includes CP violation in the Hamiltonian):

$$|K_S\rangle = \frac{N_S}{\sqrt{2}}((1+\epsilon_S)|K^0\rangle + (1-\epsilon_S)|\bar{K}^0\rangle)$$
$$\approx |K_1\rangle \qquad \text{for} \quad \epsilon_S \to 0 \tag{1.1a}$$

$$|K_L\rangle = \frac{N_L}{\sqrt{2}}((1+\epsilon_L)|K^0\rangle - (1-\epsilon_L)|\bar{K}^0\rangle)$$
$$\approx |K_2\rangle \qquad \text{for} \quad \epsilon_L \to 0 \tag{1.1b}$$

The $|K_1\rangle$ and $|K_2\rangle$ (or $|K_S\rangle$ and $|K_L\rangle$) are defined to be eigenstates of the complex "effective" Hamiltonian matrix $H = M - (i/2)\Gamma$. These vectors, and not the $|K^0\rangle$, $|\bar{K}^0\rangle$, are assumed to represent the particle states, with the justification that "since we should properly reserve the name particle for an object with unique lifetime, K_1 and K_2 are the true particles" (Pais, 1990). This effective Hamiltonian we call H if the CP-violating term H_{sw} is left aside, and we call it

$$\mathrm{H} = H + H_{sw} = H_0 + H_w + H_{sw} = H_0 + \mathrm{H}_{\mathrm{int}} \qquad \text{with} \quad [H_{sw}, CP] \neq 0 \tag{1.2}$$

if *s*ome *w*eak additional *CP*-violating interaction Hamiltonian H_{sw} (not necessarily superweak) is included. H is assumed to be normal (i.e., M, Γ are Hermitian and commute, which excludes Jordan blocks), and thus it is diagonalizable with eigenvalues given by

$$H|K_1^{0-}\rangle = \left(m_1 - i\frac{\gamma_1}{2}\right)|K_1^{0-}\rangle; \qquad H|K_2^{0-}\rangle = \left(m_2 - i\frac{\gamma_2}{2}\right)K_2^{0-}\rangle \tag{1.3a}$$

and similarly for H

$$\mathrm{H}|K_S^-\rangle = \left(m_S - i\frac{\gamma_S}{2}\right)|K_S^-\rangle; \qquad \mathrm{H}|K_L^-\rangle = \left(m_L - i\frac{\gamma_L}{2}\right)|K_L^-\rangle \tag{1.3b}$$

The time evolution for $t \geq 0$ is given by

$$\begin{aligned} e^{-iHt}|K_1^{0-}\rangle &= e^{-im_1 t}e^{-(\gamma_1/2)t}|K_1^{0-}\rangle \\ e^{-iHt}|K_2^{0-}\rangle &= e^{-im_2}e^{-(\gamma_2/2)t}|K_2^{0-}\rangle \end{aligned} \tag{1.4a}$$

or, if H is the Hamiltonian, by

$$\begin{aligned} e^{-i\mathrm{H}t}|K_S^-\rangle &= e^{-i(m_S-(i\gamma_S/2))t}|K_S^-\rangle \\ e^{-i\mathrm{H}t}|K_L^-\rangle &= e^{-i(m_L-(i\gamma_L/2))t}|K_L^- \end{aligned} \tag{1.4b}$$

The entire effective theory of the neutral K-meson system then takes place in this 2-dimensional space $\mathcal{H}_2$ spanned by the eigenvectors (1.3). An arbitrary coherent mixture (superposition) in the K^0-beam is given by

$$\phi^{\mathrm{eff}}(0) = |K_S^-\rangle a_S + |K_L^-\rangle a_L \propto |K_L^-\rangle + \rho|K_S^-\rangle \tag{1.5}$$

It has the time evolution

$$\begin{aligned} \phi^{\mathrm{eff}}(t) &= e^{-i\mathrm{H}t}\phi^{\mathrm{eff}} \\ &= e^{-im_S t}e^{-(i\gamma_S/2)t}|K_S^-\rangle a_S + e^{-im_L t}e^{-(i\gamma_2/2)t}|K_L^-\rangle a_L \end{aligned} \tag{1.6}$$

[or a corresponding expansion in terms of K_1, K_2, with a_1, a_2, if (1.3a) and (1.4a) hold]. The $|K_S\rangle$ and $|K_L\rangle$ in the coherent (pure) beam state (1.5) are conventionally expressed in terms of $|K^0\rangle$ and $|\overline{K}^0\rangle$ by (1.1), though the $|K^0\rangle$, $|\overline{K}^0\rangle$ and the $|K_S^-\rangle$, $|K_L^-\rangle$ belong to different Hamiltonians H_0 and H (or H_0 and H for $H_{sw} = 0$), respectively. The $|K^0\rangle$ and $|\overline{K}^0\rangle$ should therefore span a space different from $\mathcal{H}_2$. We indicate this difference between these two kinds of vectors by the superscript $^-$ in $|K_2^-\rangle$, etc. We shall discuss this notation in more detail in Section 3 below. An incoherent mixture, which

one usually encounters experimentally in the initial K^0 beam[2] [or which arises from a pure state if one uses a theory based on a Liouville equation for the neutral K-system (Huet and Peskin, 1995; Ellis *et al.*, 1984, 1992, 1995)], is described by a density matrix or a statistical operator in the space $\mathcal{H}_2$.

The quantity that one always considers (Kleinknecht, 1989) is the instantaneous decay rate of the K^0 state at time t into the detected decay channel c. This transition rate is proportional to $2\pi\rho_c|\langle c|\mathrm{H}_{\mathrm{int}}|\phi(t)\rangle|^2$, where $c = \pi^+\pi^-$ or $\pi^0\pi^0$ or any other decay channel. ρ_c is the phase space factor (density of states in channel c) and $|c\rangle$ is the eigenvector of the interaction-free Hamiltonian H_0. The amplitude of this rate is given according to (1.6) by

$$\langle\pi\pi|\mathrm{H}_{\mathrm{int}}|\phi^{\mathrm{eff}}(t)\rangle = \langle\pi\pi|\mathrm{H}_{\mathrm{int}}|K_S^-\rangle a_S e^{-im_S t}e^{-(\gamma_L/2)t} + \langle\pi\pi|\mathrm{H}_{\mathrm{int}}|\mathrm{K}_L^-\rangle a_L e^{-im_S t}e^{-(\gamma_L/2)t} \tag{1.7}$$

One usually considers the ratio

$$R(t) \equiv \frac{|\langle\pi\pi|\mathrm{H}_{\mathrm{int}}|\phi^{\mathrm{eff}}(t)\rangle|^2}{|\langle\pi\pi|\mathrm{H}_{\mathrm{int}}|K_S\rangle|^2} \tag{1.8}$$

for which one obtains from (1.7)

$$R(t) = |a_S|^2 e^{-\gamma_S t} + |a_L\eta|^2 e^{-\gamma_L t} + 2|a_S| \cdot |a_L| \cdot |\eta| e^{-(\gamma_L+\gamma_S)t/2}\cos(\Delta m t + \varphi) \tag{1.9}$$

where

$$\Delta m = m_L - m_S$$

and where

$$\eta = \frac{\langle\pi\pi|\mathrm{H}_{\mathrm{int}}|K_L^-\rangle}{\langle\pi\pi|\mathrm{H}_{\mathrm{int}}|K_S\rangle} = |\eta|e^{-i\varphi}$$

$$\varphi = \arg a_S - \arg(\eta a_L) \tag{1.10}$$

If $|\phi^{\mathrm{eff}}\rangle$ is $|K^0\rangle$, then $a_S = a_L = 1/\sqrt{2}$. (If ϕ^{eff} is a coherent mixture behind a regenerator, then $a_S = \rho$; $a_L = 1$.) For large values of $t \approx 20\tau_S = 20(1/\gamma_S)$ ($\tau_S = 0.893 \pm 0.001 \times 10^{-10}$ sec), only the second term in $R(t = 20\tau_S)$ of (1.9) does not have a factor of $e^{-\gamma_S t/2} \approx 10^{-5}$ or smaller, so $R(t)$ is given by

$$R(t = 20\tau_S) = \frac{1}{2}|\eta|^2 e^{-\gamma_L t}, \qquad \gamma_L/\gamma_S \approx 1.72 \times 10^{-3} \tag{1.11}$$

If in (1.2) H_{sw} would be zero and if in (1.1a)

[2]The real experimental situation is more complicated, since the initial neutral K meson is usually not a pure K^0 state, but is an incoherent mixture of K^0 and $\bar{K}^0$ due to the strangeness conservation in the production mechanism (Bohm *et al.*, 1969).

$$|K_L\rangle = |K_2\rangle \text{ with } CP|K_2\rangle = -|K_2\rangle,$$

then (due to $CP|\pi\pi\rangle = +|\pi\pi\rangle$) $\langle\pi\pi|\mathrm{H}_{\mathrm{int}}|K_L\rangle \to \langle\pi\pi|H_w|K_2\rangle = 0$, i.e., η in (1.11) should be zero. Experimentally, however, one observes $R(t = \text{large}) \neq 0$ [Princeton effect (Christenson et al., 1964)]. This is explained by the existence of an H_{sw} with the properties of (1.2), and by the decaying particle states $|K_L\rangle$ and $|K_S\rangle$ not being the CP eigenstates $|K_2\rangle$ and $|K_1\rangle$ respectively.

The two complex parameters $\eta_{+-} = |\eta_{+-}|e^{i\varphi_{+-}}$ given by (1.10) for $\langle\pi\pi| = \langle\pi^+\pi^-|$ and η_{00} given by (1.10) for $\langle\pi\pi| = \langle\pi^0\pi^0|$, are the observable quantities in terms of which one usually expresses the experimental data ascribed to CP violation. The latest experimental data, which may (NA31; Barr *et al.*, 1993) or may not (E731; Gibbons *et al.*, 1993) indicate direct CP violation: ($\langle\pi\pi|\mathrm{H}_{\mathrm{int}}|K_2^-\rangle \neq 0$, $\eta_{+-} \neq \eta_{00}$), give the following values for the CP-violation parameters:

$$|\eta_{+-}| = (2.269 \pm 0.023)10^{-3}, \qquad \phi_{+-} = 44.3° \pm 0.8° \qquad (\eta_{00} \approx \eta_{+-}) \tag{1.12}$$

Inserting this into (1.11), we obtain the following experimental value for $R(t)$:

$$R(t = 20\tau_s) \approx \frac{1}{2}|2.27 \times 10^{-3}|^2 \times 0.966 \approx \frac{1}{2}|2.23 \times 10^{-3}|^2 \tag{1.13}$$

We shall use this number for the phenomenological analysis in Section 4.

2. THE RIGGED HILBERT SPACE FORMULATION OF QUANTUM THEORY

Vectors with the properties (1.4) and (1.3) have of course no place in the Hilbert space of the standard quantum theory, where all vectors have a unitary time evolution and where self-adjoint Hamiltonians cannot have complex eigenvalues. If, on the other hand, one forces the neutral K vectors into the Hilbert space, then one derives all kinds of mathematical consequences for which there exist no experimental evidence, like deviations from the exponential decay law, and vacuum regeneration of K_S from K_L (Khalfin, 1972, 1994). The resolution of these incongruities is, of course, that the two-dimensional space $\mathscr{H}_2$ of the decaying neutral K-meson system cannot be contained in the Hilbert space. But we shall show that $\mathscr{H}_2$ is contained in the dual space $\Phi^\times$ of a rigged Hilbert space (or Gelfand triplet) $\Phi \subset \mathscr{H} \subset \Phi^\times$.

A theory of resonance scattering and decay has been developed over the past two decades (Bohm, 1978, 1981, 1993, 1995; Gadella, 1985a,b, 1984; Antoniou, 1992; Antoniou and Prigogine, 1993; Antoniou and Tasaki, 1993; Bohm *et al.*, 1994, 1995) which uses the rigged Hilbert space (RHS)

formulation of quantum mechanics. (Roberts, 1966; Bohm, 1967; Antoine, 1969a; Melsheimer, 1974).

Whereas in the Hilbert space the solutions of the Schrödinger equation can have only unitary time evolution and the eigenvalues of self-adjoint operators can only be real, the RHS formulation allows for a greater variety of solutions and for new initial and boundary conditions with a preferred direction of time (irreversibility). In this formulation, heuristic notions like Dirac kets $|E\rangle$,Gamow's (exponentially decaying "state") vectors $|E - i(\Gamma/2)\rangle$, and Peierls purely outgoing boundary conditions (Gamow, 1928; Peierls, 1955; Garcia-Calderon and Peierls, 1976; Hernandez and Mondragon, 1984) can be given a unified and mathematically meaningful foundation. Vectors with the properties (1.3) and (1.4) are well defined in the RHS formulation as generalized eigenvectors [see Appendix A, equation (A.6)] of a self-adjoint Hamiltonian (semibounded, essentially-self-adjoint operator H in an infinite-dimensional space), and the time evolution of these vectors is given by an irreversible semigroup generated by the Hamiltonian. These new vectors, which can also be obtained from a resonance pole of the analytically continued S-matrix and consequently have a Breit–Wigner energy distribution, have been called Gamow vectors.

Since this paper addresses a physics problem, we do not give here the precise mathematical definition of the RHS in general and of the particular spaces Φ, Φ_+, Φ_- and their topological duals (space of continuous antilinear functionals) $\Phi^\times$, $\Phi^\times_+$, $\Phi^\times_-$ which we shall use in this paper. In Appendix A we give a brief and casual description of the RHS. For the calculations in this paper we adopt the modus operandi of a physicist and do not worry about the precise mathematical definitions of $\mathcal{H}$ or Φ or Φ_+, Φ_-, $\Phi^\times_+$, etc. All these spaces are different topological completions of the same pre-Hilbert space Ψ [i.e., a linear space Ψ with "scalar product" denoted by (ψ, F) or by $\langle\psi|F\rangle$]. Here we use the algebraic space Ψ and some additional rules which can be justified by the mathematics of the RHS. These additional rules include the well-known rules of the Dirac bra-and-ket formalism, some mathematical properties of the Gamow vectors [e.g., those given by (1.3) and (1.4)], and in particular some basis vector representations which were not part of the Dirac formalism (e.g., the so-called "complex spectral representation" of Appendix B). These rules can only be justified by the full RHS theory. We shall simply introduce these rules as needed, while referring the reader to the literature (Bohm, 1978, 1981, 1994; Gadella, 1983a,b, 1984; Antoniou, 1992; Antoniou and Prigogine, 1993; Antoniou and Tasaki, 1993; Bohm *et al.*, 1996; Bohm and Gadella, 1989) for their justification.

However, since the distinction between the spaces Φ_+ and Φ_- will be of physical importance, we need to explain some of their mathematical differences. This is done using their mathematical "realizations." One often

says in mathematics that an abstract (linear topological) space is "realized" by a function space if there exists a correspondence between each vector $\phi \in \Phi$ and an element (or elements) of the space of functions (probably in mathematics functions are more "real" than vectors). The basis vector expansions of Appendix B are examples of mathematical "realizations." In this manner the space Φ is represented or "realized" by the standard test function space (Schwartz space). The space Φ_+ (and Φ_-) is "realized" by the subspace of Hardy class functions, where the $+$ $(-)$ refers to analyticity in the upper (lower) half-plane of the second sheet of the complex energy surface, (for mathematical details see Bohm and Gadella, 1989, Bohm, 1994; Bohm *et al.*, 1996).

In physics the abstract mathematical objects are realized by physical objects. Thus a physicist's "realization" of the linear spaces Φ_+, Φ_- and their duals are by quantum physical objects like states and observables. The standard quantum theory uses the same Hilbert space for both states and observables.

In contrast, distinct initial-boundary conditions for state vectors ϕ^+ (e.g., in-states ϕ^+ of a scattering experiment) and observables $|\psi^-\rangle\langle\psi^-|$ (e.g., so-called out-states ψ^- of a scattering experiment) lead to two different rigged Hilbert spaces (Bohm, 1993, 1995; Bohm *et al.*, 1994, 1995):

$$\phi^+ \in \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \qquad \text{for in-states of a scattering experiment which are prepared by a preparation apparatus, e.g., an accelerator} \tag{2.1}$$

$$\psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \qquad \text{for observables or out-states which are measured by a registration apparatus, e.g., a detector} \tag{2.2}$$

The Hilbert space $\mathcal{H}$ is the same in both RHS (2.1) and (2.2). However, Φ_+ and Φ_- are different, but they have more than the zero vector in common, $\Phi_- \cap \Phi_+ \neq \{0\}$; in fact, $\Phi_- \cap \Phi_+$ is in general infinite dimensional. To use different mathematical spaces for states (in-states and observables (so-called out-"states") is one of the new features of the RHS formulation of quantum mechanics.

In (2.1) Φ_- describes the possible state vectors experimentally given by the preparation apparatus (e.g., ϕ^{in} or ϕ^+ of a scattering experiment) and in (2.2) Φ_+ describes the possible observables (e.g., $|\psi^{out}\rangle\langle\psi^{out}|$ or $|\psi^-\rangle\langle\psi^-|$ of a scattering experiment) experimentally specified by the detector. The $|F^\pm\rangle \in \Phi_\pm^\times$ represent quantities connected with the microphysical system (e.g., "scattering states $|E^\pm\rangle$ or decaying states $|E - i\Gamma/2^\pm\rangle$). The superscripts for the vectors $\phi^+ \in \Phi_-$ and $\phi^- \in \Phi_+$ and the superscripts in $|E^\pm\rangle \in \Phi^\times$ for the eigenkets of the Hamiltonian $H = H_0 + H_{\text{int}}$ refer to the standard

notation of scattering theory, where ϕ^+ represents the in-states and Ψ^- represents the out-observables (also called out-states). The vectors $|E^{\pm}\rangle$ are related by the Lippmann–Schwinger equations to the eigenkets $|E\rangle$ of H_0, and the superscripts of the eigenkets $|E - i\Gamma/2^-\rangle \in \Phi_+^\times$ and $|E + i\Gamma/2^+\rangle \in \Phi_-^\times$ of the (essentially self-adjoint, semibounded) Hamiltonian H with complex eigenvalue $(E \mp i\Gamma/2)$ are an extension of the labels in $|E^{\mp}\rangle$. (The antithetical subscripts for the spaces have their origin in mathematicians' notation for the Hardy class functions.) In this paper we shall only use the Gamow vectors $|E - i\Gamma/2^-\rangle$ which have the property (1.4) describing exponentially decaying microphysical objects.

3. THE COMPLEX TWO-DIMENSIONAL HAMILTONIAN OF THE NEUTRAL *K*-SYSTEM AS A TRUNCATION OF THE EXACT HAMILTONIAN IN THE RHS

The neutral K meson is produced in an inelastic scattering experiment. We discuss here the case that this inelastic scattering process produces pure K^0 states, as, e.g., in the reaction (see footnote 2)

$$\pi^- p \to \Lambda K^0 \to \Lambda\pi\pi \tag{3.1}$$

The principles of a scattering experiment applied to this process are depicted in Fig. 1. The scattering experiment consists of a preparation part and a registration part. A meson beam π^- (i.e., in the $II_3 y = 1, -1, 0$ state), which is prepared as a ϕ^{in} before the interaction with the target B, evolves in the interaction region as a $\phi^+ \in \Phi_-$. Due to the interaction it makes a transition into the prepared state of the K^0 flavor, described according to (2.1) by a $\phi^+_{y=1} \in \Phi_-$. Thus the part of the experiment which prepares the state $\phi^+_{y=1}$ consists of the apparatus for the preparation of the π^- beam and the strong interaction with the baryon target B (changing the target state from p to Λ). The registration part of the experiment determines the so-called out-"state" ψ^- which is registered outside the interaction region as, e.g., either $\psi^{\text{out}} = |\pi^+\pi^-\rangle$ or $\psi^{\text{out}} = |\pi^0\pi^0\rangle$. Its principal component is the $\pi^+\pi^-$ and/or $\pi^0\pi^0$ detector. According to (2.2), the out-state, which is actually an observable $|\psi^-\rangle\langle\psi^-|$, satisfies $\psi^- \in \Phi_+$. In the conventional formulation of scattering theory in the Hilbert space, the in-state ϕ^+ (and ϕ^{in}) as well as the out-observable ψ^- (and ψ^{out}) can be any vector of the Hilbert space $\mathcal{H}$. In reality the ϕ^+ and ψ^- are subject to different conditions, namely, initial conditions for ϕ^+ and final conditions for ϕ^-. This is described in the new RHS quantum theory by using different mathematical conditions, $\phi^+ \in \Phi_-$ and $\psi^- \in \Phi_+$, as shown in equations (2.1) and (2.2), respectively.

We now consider the state of the meson at times (in the rest frame of the K^0) $t \geq 0$. Here $t = 0$ is the time before which the preparation of the

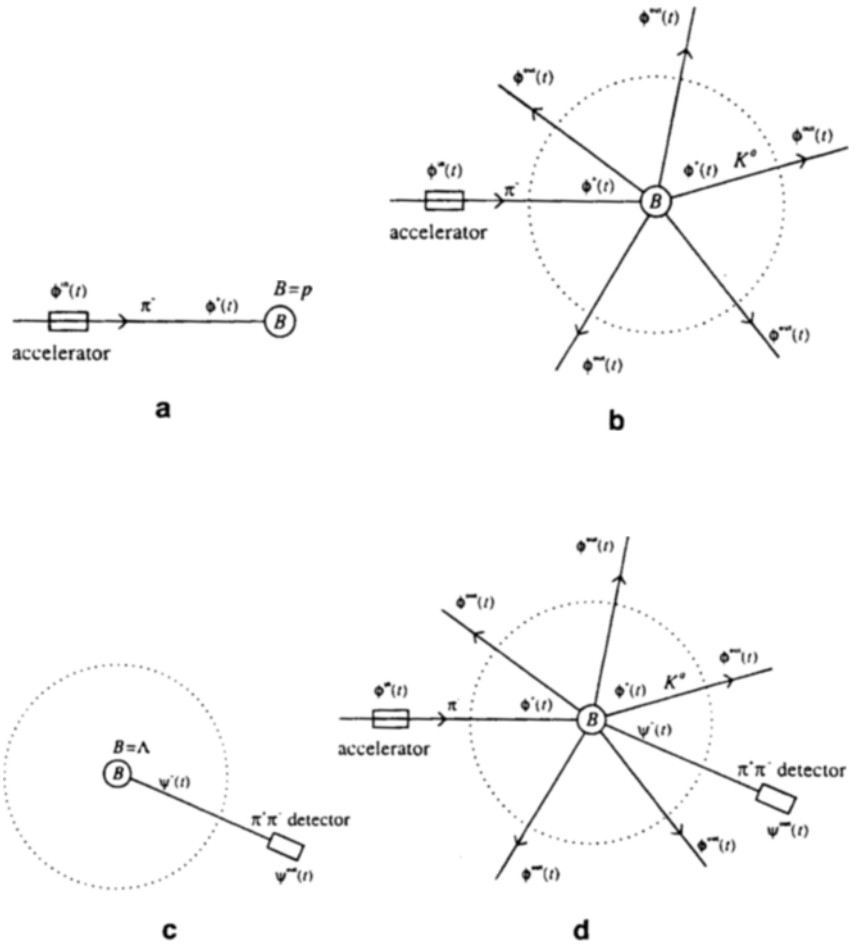

Fig. 1. The definition of the spaces Φ_- (in-states) and Φ_+ (out-observables). The preparation and registration procedure for quantum systems (Ludwig, 1983/1985) is applied to the inelastic scattering experiment for production and decay of the neutral K-system. (a) The preparation of the π^- state. (b) The preparation apparatus of the K^0 system, which consists of the preparation of π^- and the interaction with the prepared baryon system B. (c) The registration apparatus which defines the out-"state" $|\psi^-\rangle\langle\psi^-|$ (observable); it principally consists of the $\pi\pi$ detector. Every arrangement for an experiment with single microsystems consists of a preparation apparatus and a registration apparatus. (d) The preparation part of panel b combined with the registration part of panel c into the experiment that measures the probability for the transition $K^0 \to \pi\pi$. Since the in-states ϕ^+ and the out-observables ψ^- are subject to different conditions, they are described in the RHS quantum theory by distinct spaces Φ_- and Φ_+, respectively.

neutral K-meson state (or in general of the ϕ^+ state) is completed and after which the registration of $\psi^{out} \leftarrow \psi^-$ begins. The time t is the proper time of the kaon, which in the actual experiment is measured by the distance d from the target position (or from the exit face of the regenerator in regeneration experiments when $\phi^+ \sim f\phi^+_{y=1} + \bar{f}\phi^+_{y=-1}$) to the decay vertex ($t = dm_K/cp$, where p is the competent of the kaon momentum along the beam line). To each decay vertex we associate a microphysical exponentially decaying state of the neutral K-system, which is represented by a Gamow vector. This exponentially decaying Gamow state is a component of $\phi^+(t)$ that has dynamically evolved in time from the original, prepared state $\phi^+ = \phi^+(t = 0)$ by the exact Hamiltonian H. (The Hamiltonian H contains all interactions, including the one responsible for the kaon decay.) Since ϕ^+ is according to (B.2) a superposition of functionals, the time evolution operator is given by the semigroup $e^{iH^\times t}$; thus[3]

$$\phi^+(t) = e^{-iH^\times t}\phi^+ \qquad \text{or} \qquad \phi^+(t) = e^{-iH^\times t}\phi^+, \qquad t \geq 0 \tag{3.2}$$

We first want to discuss a theory in which the CP-violating H_{sw} is assumed to be zero and for which the exact Hamiltonian is denoted by H. Then $[H, CP] = 0$ and we choose H and CP as the complete system of commuting observables (c.s.c.o.). We ignore all other observables for the neutral K system except for the energy operator H, the weak-interaction-free energy operator H_0; the hypercharge $\hat{Y}$, and the discrete symmetries like CP. [This means we are always working in the K-meson rest frame where we have $(\vec{p}_K, j_3, j^\pi) = (\vec{0}, 0, 0^-)$, which we do not write).] In place of the c.s.c.o. H, CP, one can also consider the c.s.c.o. H_0, $\hat{Y}$. Besides these two, there is still another c.s.c.o.: H_0 and CP, which we, however, do not want to consider. The operators H and $\hat{Y}$ do not form a c.s.c.o.

Now we shall make use of some exact results in Appendix B. We want to contrast the two basis systems given by the nuclear spectral theorem (B.1) and by the complex basis vector expansion (B.2) for the c.s.c.o. H, CP. The system of basis vectors is denoted, respectively, by

$$|E, cp^-\rangle \qquad \text{for} \qquad 0 \leq E < \infty \tag{3.3a}$$

[3]The notation $H^\times$ is the notation of (A.7). Since H is self-adjoined, we usually drop the $^\times$, i.e., we use the notation of (A.8). The important feature of (3.2) is that $t \geq 0$, i.e., that we want to consider $\phi^+(t)$ for $t \geq 0$ and then $\phi^+ \in \Phi_-$ is to be understood as a functional over Φ_+, i.e., $\phi^+(t) \in \Phi^\times_+$; note $\Phi^\times_+ \supset \Phi_- \ni \phi^+$. To emphasize this, we retain the mathematically precise notation $e^{-iH^\times t} \equiv (e^{iHt})^\times$ in (3.2). As a consequence of the fact that $\phi^+(t) \in \Phi^\times$ for $t > 0$, it follows that one can take only bra-kets of $\phi^+(t)$ with ψ^- = out-observables (like, e.g., $\psi^- = |\pi\pi\rangle$), which means that $\langle\psi^-|\phi^+(t)\rangle$ is well defined for $t > 0$ but $\langle\tilde{\phi}^+|\phi^+(t)\rangle$ is not well defined if $\tilde{\phi}^+$ is any element of Φ_-. This is sufficient for the physical problem, since only $|\langle\psi^-|\phi^+(t)\rangle|^2$ has a meaning, namely the probability for a transition from the state ϕ^+ into the decay products ψ^-.

and by

$$|z_i, cp^-\rangle \text{ for } z_i = z_{K_1}, z_{K_2}; \quad \text{and} \quad |\omega, cp^+\rangle \text{ for } -\infty_{II} < \omega \leq 0 \quad (3.3b)$$

We also shall consider the eigenvectors of the c.s.c.o. H_0, Y, which we denote by

$$|E, y\rangle \qquad \text{for} \qquad 0 \leq E < \infty \quad (3.4)$$

The *missing* superscript $^-$ in (3.4) indicates that $|E, y\rangle$ are eigenvectors of H_0 and not of H (this latter notation is not in agreement with the standard notation of scattering theory, where $|E\rangle$ would denote the eigenvector of $H_{00} = H_0 - H_{\text{strong}}$).

The vectors

$$|E = m_K, y = \pm 1\rangle \qquad \text{correspond to the usual} \quad |K^0\rangle, |\bar{K}^0\rangle \quad (3.5)$$

while the vectors

$$|z_1, cp = +, ^-\rangle, \qquad |z_2, cp = -, ^-\rangle \qquad \text{correspond to the usual} \quad |K_1\rangle\ |K_2\rangle \quad (3.6)$$

Therefore we also use the notation

$$|K_i, ^-\rangle = |z_i, cp = -(-1)^i, ^-\rangle, \qquad z_i = m_{K_i} - i\frac{\gamma_i}{2}, \qquad i = 1, 2 \quad (3.6a)$$

[up to an arbitrary "normalization" constant; cf. (3.18) below and footnote 6]. This notation also explains why already in equations (1.3) and (1.4) we employed the unusual notation $|K_1^-\rangle$ and $|K_2^-\rangle$ with a superscript $^-$, in contrast to the usual notation $|K_1\rangle$ and $|K_2\rangle$. The $|K_i^-\rangle$ are the exact generalized eigenvectors of the self-adjoint, semibounded Hamiltonian H with complex eigenvalues $z_i = m_i - i\gamma_i/2$, i.e., they are the Gamow kets of the neutral kaon system. The Gamow kets satisfy *exactly* the (exponential) time evolution equation (1.4), with H being the infinite-dimensional self-adjoint Hamiltonian operator and not a complex two-dimensional matrix H^{eff}. But (1.4) [and also (1.3)] is to be understood as a functional equation over the space Φ_+:

$$\langle e^{Ht}\psi^-|z_i^-\rangle \equiv \langle \psi^-|e^{-iH^\times t}|z_i^-\rangle = e^{-i(m_i - i(\gamma_i/2))t}\langle \psi^-|z_i^-\rangle$$
$$\text{for all} \quad \psi^- \in \Phi \quad \text{and} \quad t \geq 0 \quad (3.7)$$

This means one can only form Dirac bra-ket of (1.4) with a ψ^- that is an element of the infinite-dimensional space Φ_+ and not in general with $\phi^+ \in \Phi$ (see footnote 3).

That the vectors ψ^- in (3.7) can only be of the space Φ_+ (i.e., representing the registered decay products, e.g., $\psi^{out} = |\pi^+\pi^-\rangle$ is not a restriction on physics, since we are interested in decay probabilities or transition rates into observed out "states" and not in arbitrary matrix elements. That $t \geq 0$ in (3.7) i.e., that the time evolution of the Gamow vectors is given by a semigroup, not a unitary group, is derived using the mathematics of the RHS (Bohm, 1978, 1981, 1993–1995, Gadella, 1983a, b, 1984, Antoniou and Prigogine, 1993; Antoniou and Tasaki, 1993) and is the appropriate restriction for decay processes (arrow of time, microphysical irreversibility).

Using the c.s.c.o. H, CP, we have two choices for the basis vector expansion:

1. Dirac basis vector expansion (B.1):

$$\phi^+ = \sum_{cp=\pm 1} \int_{(2m_\pi)}^{\infty} dE\, |E, cp^+\rangle b_{cp}(E) \tag{3.8}$$

where the expansion coefficients (energy wave function) are well-behaved functions along the cut in the physical sheet $2m_\pi < E < \infty$

$$b_{cp}(E) = \langle {}^+E, cp|\phi^+\rangle \in \mathcal{S} \quad \text{(Schwartz space)} \tag{3.9}$$

If ϕ^+ is a state of the neutral kaon system, then $b_{cp}(E)$ are presumed to be peaked at $E = m_{K_1}$ for $cp = 1$ and at $E = m_{K_2}$ for $cp = -1$. These functions may in the neighborhood of the energy value $E = m_{K_1}$ come close to being Breit–Wigner amplitudes (with widths γ_i).

The expansion (3.8) has its *analogue* in Hilbert space quantum mechanics. Hilbert space quantum mechanics amounts to the assumption that the energy wave functions satisfy the conditions

$$b_{cp}(E) \in L^2 \qquad \text{and} \qquad Eb_{cp}(E) \in L^2 \quad \text{(Lebesgue square integrable)} \tag{3.10}$$

Then $b_{cp}(E)$ cannot be a Breit–Wigner amplitude. Further, the assumption (3.8) with (3.9) or (3.10) causes the well-known problems of deviations from the exponential decay law in general[4] and some additional problems specifically for the neutral K-system (Khalfin, 1972, 1994).

If the prepared state ϕ^+ is a pure strangeness $= +1$ state $\phi^+ = \phi^+_{y=1}$, which would usually be denoted by $\phi^+_{y=1} = |K^0\rangle$, then for that $\phi^+_{y=1}$ the wave

[4]Deviations from the exponential law for large values of t follow from the finite lower limit $E = 2m_\pi$ in the integral (3.8) (in contrast, the integral representation of the Gamow vectors extends over $-\infty_{II} < E < +\infty$, where the negative E are on the second energy sheet of the S-matrix). Deviations for small values of t follow from the condition that ϕ^+ be in the domain of H [second equation (3.10); Fonda *et al.* (1978), and references therein].

functions in (3.8) would satisfy[5]

$$b_{cp=-1}(E) = b_{cp=+1}(E) \equiv \frac{b(E)}{\sqrt{2}} \tag{3.11}$$

This means that a prepared state vector with a definite hypercharge $y = +1$ or -1 has the Dirac basis vector expansion

$$\phi_y^+ = \int_{2m_\pi}^{\infty} dE(|E, cp = +1, {}^+\rangle + \text{sign}(y)|E, cp = -1, {}^+\rangle) \frac{b(E)}{\sqrt{2}} \tag{3.12}$$

If one defines

$$\text{“}|K_{1,2}^0\rangle\text{”} = \int_{2m_\pi}^{\infty} dE|E, cp = +1, -1, {}^+\rangle b(E) \tag{3.13}$$

and uses the notation

$$\phi_{y=1}^+ = |K^0\rangle, \qquad \phi_{y=-1}^+ = |\bar{K}^0\rangle \tag{3.14}$$

then (3.12) is the standard expression

$$|K^0\rangle = \frac{1}{\sqrt{2}}(\text{“}|K_1^0\rangle\text{”} + \text{“}|K_2^0\rangle\text{”}), \qquad |\bar{K}^0\rangle = \frac{1}{\sqrt{2}}(\text{“}|K_1^0\rangle - \text{“}|K_2^0\rangle\text{”}) \tag{3.15}$$

However, the vectors “$K_{1,2}^0\rangle$” do not give the Lee–Oehme–Yang theory, because they cannot have the properties (1.4a) and (1.3a) that one demands of the usual $|K_{1,2}^0\rangle$. Indeed, if one defines $|K_{1,2}^0\rangle$ by (3.13) and imposes the precise conditions (3.10) given by the Hilbert space formulation, then one arrives at all kinds of “*CP*-violation problems of the exact (Hilbert space) quantum theory” (Khalfin, 1972, 1994).

We shall not use Hilbert space quantum theory here and thus not make the assumption (3.10). Instead we will use the conditions (2.1), which in terms of the wave function $b_{cp}(E) = \langle {}^+E, cp|\phi^+\rangle$ means $b_{cp}(E) \in \mathcal{S} \cap \mathcal{H}_-$ [Hardy class function from below (Bohm and Gadella, 1989; Bohm, 1994)].

[5]Though (3.11) looks suggestive because of (3.15) it is not obtained without additional explanations: Since $\{CP, Y^{\text{op}}\} = 0$, the hypercharge operator Y^{op} changes the eigenvalues $cp = \pm 1$ of the operator CP; thus

$$Y^{\text{op}}|E, cp^+\rangle = \eta_{cp}(E)|E'(-cp)^+\rangle$$

where the phase factor $\eta_{cp}(E)$ can be chosen to be unity, fixing the relative phases of the wave function $b_{+1}(E)$ and $b_{-1}(E)$ of (3.11) in the standard way of (3.15). We also choose the phases such that $CP\phi_y^+ = \phi_{-y}^+$ for the vector in (3.12). However, since $[H, Y^{\text{op}}] \neq 0$ (due to H_w in $H = H_0 + H_w$), Y^{op} also shifts the energy (i.e., the invariant mass of the decay products of $K_{1,2}^0$) slightly. Thus, more accurately one would have to take in place of (3.11): $b_+^y(E) = \text{sign}(y)b_-^y(E')$, where $E' \approx E$. Since we do not want to use the Dirac basis vector expansion (3.8), we need not pursue this point any further.

Then one can also use the Dirac basis vector expansion (B.1) to obtain (3.8) for $\phi^+ \in \Phi_-$. But as an alternative to (B.1) one has a second choice:

2. The complex basis vector expansion (B.2):

$$\phi^+ = |K_1^{0-}\rangle b_1 + |K_2^{0-}\rangle b_{-1} + F^-_{cp=+1} + F^-_{cp=-1} \tag{3.16}$$

Here we have defined

$$F^-_{cp} = \int_{(2m_\pi)}^{-\infty_{II}} dE\, |E, cp^-\rangle b_{cp}(E) \tag{3.17}$$

which we call the background term. And we have defined the (differently normalized[6]) Gamow vectors

$$|K_i^{0-}\rangle = |z_i, cp = -(-1)^i, {}^-\rangle\sqrt{2\pi\gamma_i}; \qquad z_i = m_i - i\frac{\gamma_i}{2} \tag{3.18}$$

or the microphysical state operator

$$|K_i^{0-}\rangle\langle {}^+K_i^0| = |\, z_i, {}^-\rangle 2\pi\gamma_i \langle {}^+z_i| \tag{3.19}$$

Since $|K_i^{0-}\rangle \in \Phi^\times_+$, the vectors defined by the r.h.s. of (17) must also be in $\Phi^\times_-$: $F^-_{cp} \in \Phi^\times_+$, because $\phi^+ \in \Phi_-$

According to the theory that underlies (B.2), the eigenvalues

$$z_i = m_i - i\frac{\gamma_i}{2}$$

in (3.18) and (3.19) are the complex energies of $K_i^0 (i = 1, 2)$. This means they are the positions of the resonance poles for the two resonances K_1^0 and K_2^0. (e.g., of the S-matrix for the scattering process with resonance formation[7]

[6]The normalization of the $|z_R^-\rangle$ has its origin in the δ-function "normalization" of the Dirac ket, $\langle {}^-E|E'{}^-\rangle = \delta(E - E')$, and in the Titchmarsh theorem. The "normalization" of the $|K_i^-\rangle$ has been chosen such that in the limit $\gamma_i/m_i \to 0$ it agrees with the normalization (unity) of the usual $|K_{1,2}^0\rangle$. Since the normalization is not relevant for our problems, we choose here the more familiar normalization, which leads to the factor $\sqrt{2\pi\gamma_i}$ in (3.18).

[7]The Gamow vectors can be defined as vectors $\in \Phi^\times$ that are associated with the resonance pole of the S-matrix in the same way as the bound-state vectors can be defined as the vectors associated with the bound-state poles of the S-matrix. Whereas in the standard Hilbert space quantum mechanics, the bound states can also be defined independently of the S-matrix, resonance states cannot. But in the RHS formulation both can be defined independently of the S--matrix, as (generalized) eigenvectors of H. Using the S-matrix definition of resonances, the associated Gamow vectors can then be shown to satisfy (1.3) and (1.4) for $t \geq 0$ [or precisely (3.7) as generalized eigenvalue equations (A.6)]. The same Gamow vectors appear also in the complex "spectral" decomposition (B.2), which is an exact representation for every very well-behaved vector $\phi^+ \in \Phi_-$, i.e., for every vector which in our interpretation (2.1) (based on the preparation → registration arrow of time) represents an in-state prepared by an experimental apparatus. Note that the existence of this preferred direction of time can be formulated as: A state must be prepared first before an observable can be measured in it. The Liouville equation of Huet and Peskin (1995) and Ellis *et al.* (1984, 1992, 1995) leads, like our semigroup, to an "arrow of time"; however their irreversibility is due to extrinsic influences, whereas in our case the time evolution is generated by the Hamiltonian of the system.

$$\pi\pi \to (K^0, \overline{K}^0) = (K_1^0, K_2^0) \to \pi\pi \tag{3.20}$$

or any other scattering processes in which K-resonance formation occurs). This value is identical with the generalized eigenvalue of H.

The expansion coefficient—i.e., the coordinates of the vector ϕ^+ along the basis vectors (3.3a) and (3.3b)—are, according to (B.2), given by

$$b_{cp=+1} = \langle {}^+K_1^0|\phi^+\rangle, \qquad b_{cp=-1} = \langle {}^+K_2^0|\phi^+\rangle = \sqrt{2\pi\gamma_2}\langle {}^+z_2|\phi^+\rangle \tag{3.21}$$

$$b_{cp}(E_{II}) = \langle {}^-E_{II}, cp|\phi^+\rangle, \qquad \langle {}^-E_{II}, cp|\phi^+\rangle = S_{cp}(E_{II})\langle {}^+E_{II}cp|\phi^+\rangle \tag{3.22}$$

where $S(E)$ is the S-matrix analytically continued to values E_{II} at the negative real axis on the second sheet [of, e.g., the scattering process (3.20) if we consider, as we shall, the decays $K^0 \to \pi\pi$].

The representation (3.16) is the special case of (B.2) if there are no bound states (i.e., no neutral K's with a mass below the $\pi^+\pi^-$ threshold) and if there are only two resonance states with $j^\pi = 0^-$. Under these assumptions the representation (3.16) is therefore exact (like the spectral theorem) for every $\phi^+ \in \Phi_-$. Whereas the vectors $|K_i^{0-}\rangle$ are well-known basis vectors, the "background integrals" (3.17) (integrals along the negative real axis of the second sheet) depend upon the dynamics H or S, and upon the prepared state ϕ^+ and are not well known.[8]

The best situation is obtained if one can prepare a state ϕ^+ such that the background term is very small. Then one has from (3.16) for any pure (i.e., coherent "mixture" or superposition) neutral kaon state

$$\phi^+ \approx |K_1^{0-}\rangle b_1, \quad +|K_2^{0-}\rangle b_{-1} \tag{3.23}$$

This means that for values of $|b_{cp}(E)| << |b_{cp}/\sqrt{\gamma_i}|$ any prepared pure neutral kaon state is approximately a superposition of $|K_1^{0-}\rangle$ and $|K_2^{0-}\rangle$.

We now shall take for the prepared state ϕ^+ a hypercharge eigenstate as produced, e.g., by the elastic scattering process (3.1)[9] (Gell-Mann and Pais hypothesis) which we call $\phi^+_{y=+1}$ (corresponding to $|K^0\rangle$) or $\phi^+_{y=-1}$ (corresponding to $|\overline{K}^0\rangle$). Then

$$\phi_y^+ = |z_1, cp = +1^-\rangle \tilde{b}_1^y + |z_2, cp = -1^-\rangle \tilde{b}_1^y + \int_{2m_\pi}^{-\infty_{II}} dE\, (|E, cp = +1^-\rangle b_1^y(E) + |E, cp = -1, {}^-\rangle b_{-1}^y(E)) \tag{3.24}$$

[8] The Hardy class functions in the half-plane $\langle {}^\pm\omega|\phi^\pm\rangle$ are already determined by their values on the positive real axis, i.e., by their physical values of energy E, but $S(E)$ depends upon the dynamics of the problem and $\langle {}^+E|\phi^+\rangle$ on the choice of the prepared state vector (preparation apparatus).

[9] In the realistic experiment (without regeneration) one does not have pure states, but an incoherent mixture. But, as a consequence of strangeness conservation in the strong production mechanism, these are incoherent mixtures of $\phi^+_{y=+}$ and $\phi^+_{y=-}$, i.e., the state is $w_1(p)\phi^{+(p)}_{y=1}\rangle\langle\phi^{+(p)}_{y=1}| + w_{-1}(p)|\phi^{+(p)}_{y=-1}\rangle\langle\phi^{+(p)}_{y=-1}|$ integrated over all values of the K-momenta p.

and using the same arguments that led to (3.11),[10] one obtains

$$\frac{1}{\sqrt{2\pi\gamma_1}}\tilde{b}_1^{y=\pm 1} = \pm b_{-1}^{y=\pm 1} \equiv \frac{b}{\sqrt{2}}, \qquad \pm b_1^{y=\pm 1}(E) = b_{-1}^{y=\pm 1}(E) \equiv \frac{b(E)}{\sqrt{2}} \tag{3.25}$$

For a hypercharge eigenstate the approximation (3.23) then becomes

$$\phi_{y=\pm 1}^{+} \approx (|K_1^-\rangle \pm |K_2^-\rangle)\frac{b}{\sqrt{2}} \tag{3.26}$$

This is the analogue of the standard expressions (3.15), except that the $|K_i^-\rangle$ in (3.26) are the Gamow vectors with the property (1.3) that have the exact time evolution (3.7) or (1.4).

We now obtain the time evolution (3.2) of the prepared state ϕ_y^+ for $t > 0$ using the representation (3.24), since for $t \geq 0$ the evolution of the Gamow vectors is known from (3.7),

$$\begin{aligned}\phi_{y=\pm 1}^{+}(t) &= e^{-iH^{\times}t}\phi_{y=\pm 1} \\ &= \frac{b}{\sqrt{2}}(e^{-im_1 t}e^{-(\gamma_1/2)t}|K_1^-\rangle \pm e^{-im_2 t}e^{-(\gamma_2/2)t}|K_2^-\rangle) \\ &\quad + \int_0^{-\infty_{II}} dE\,(\pm|E, cp = +1^-\rangle + |E, cp = -1^-\rangle)e^{-iEt}\frac{b(E)}{\sqrt{2}}\end{aligned} \tag{3.27}$$

Note that according to (3.27), $e^{-iH^{\times}t}$ cannot transform from the background term $(F_{+1}^- + F_{-1}^-)$ to $|K_1^-\rangle$ or $|K_2^-\rangle$ and it cannot transform from $|K_2^-\rangle$ to $|K_1^-\rangle$. This means our theory does not predict anything that could be interpreted as "vacuum regeneration of K_1 (or K_S) from K_L" (Khalfin, 1972, 1994), which is also not observed.

The approximation of a prepared state for very small background term, $|b(E)\sqrt{\gamma_i}/b| << 1$, given by (3.26), has the time evolution

$$\phi_{y=\pm}^{+}(t) \equiv e^{-iH^{\times}t}\phi_{y=\pm 1}^{+} = \frac{b}{\sqrt{2}}(e^{-im_1 t}e^{-(\gamma_1/2)t}|K_1^-\rangle \pm e^{-im_2 t}e^{-(\gamma_2/2)t}|K_2^-\rangle) \tag{3.28}$$

[10] Due to $[H, Y^{op}] \neq 0$ the operator Y^{op} is not a symmetry group generator, but a spectrum-generating group generator changing not only the eigenvalue cp of CP, but also the eigenvalue z_i of H:

$$Y^{op}|z_1, cp = 1,^-\rangle = \eta_1|z_2, cp = -1,^-\rangle$$

$$Y^{op}|z_2, cp = -1,^-\rangle = \eta_{-1}|z_1, cp = -1,^-\rangle$$

where η are phase factors (which can be chosen $+1$ by absorbing then into the b_i: $b_1 \to \eta_1 b_1$, $b_{-1} \to \eta_{-1}b_{-1}$). These relations are needed to establish (3.25). See also footnote 5.

This means that the RHS quantum theory reproduces to a certain extent the Lee–Oehme–Yang theory, and if ϕ^+ can be prepared such that the background term $|b_{cp}(E_{II})|$ is negligibly small, then the Lee–Oehme–Yang theory emerges as its approximation. Conversely, the worthiness of the Lee–Oehme–Yang theory can be taken as a measure of how small the background term $F_{cp}^- \in \Phi_+^\times$ in (3.16) and (3.24) can be made if the K^0-state $\phi_y^+ \in \Phi_-$ is suitably prepared.

Thus the "complex spectral" resolution (B.2) of the RHS formulation of quantum mechanics chooses the basis system in $\phi_+^\times$ such that the two-dimensional space $\mathcal{H}_2$ of the standard neutral kaon model is spanned by the generalized basis vectors $|K_1^-\rangle, |K_2^-\rangle \in \Phi_+^\times$. One does not have to make any special assumptions about a complex effective Hamiltonian. The Hamiltonian H (precisely its closure $H^\dagger$) *is just required to have the standard properties: It is a self-adjoint operator, bounded from below.*

The two-dimensional matrix H^{eff} emerges as the matrix of this infinite-dimensional operator H in the two-dimensional subspace $\mathcal{H}_2 \subset \Phi_+^\times$. Thus the RHS formulation justifies the effective Lee–Oehme–Yang theory—in contrast to the exact Hilbert space formulation, which contradicts it (Khalfin, 1972, 1994). The RHS formulation also gives the semigroup ($t \geq 0$) time evolution of the Kaon decay, (3.27), (3.28), which is one of the many manifestations of microphysical irreversibility. Furthermore, for the kaon system in particular, it predicts the background term, which, however small, must be different from zero (because $\phi^+ \in \Phi_-$ cannot be a superposition of two vectors $|K_1^-\rangle, |K_2^-\rangle \in \Phi_+^\times$).

The existence of this background term in (3.27) has some significant consequences, which may or may not be of practical importance since $|\sqrt{\gamma}b(E)/b| << 1$ could be too small to be observed. However, since in the case of CP violation and direct CP violation one is talking of effects of the order of 10^{-3} or 10^{-6} and since one also discusses CPT violations and violations of microphysical quantum coherence (Maiani, 1992; LeGac, 1996; Huet and Peskin, 1995; Ellis *et al.*, 1984, 1992, 1995), which are orders of magnitude smaller than 10^{-3}, a discussion of possible effects from the background terms is warranted.

4. LONG-TIME 2π DECAYS OF NEUTRAL K WITHOUT CP-VIOLATING HAMILTONIAN

The quantity that is measured in the neutral K experiments (Gibbons, 1993, esp. Sections 9.2, 9.3; Geweniger *et al.*, 1974; Christenson *et al.*, 1974; Barr *et al.*, 1993; Gibbons *et al.*, 1993) is the instantaneous decay rate of the K^0's into $\pi\pi$ ($\pi^+\pi^-$ or $\pi^0\pi^0$). In analogy to the instantaneous transition rate of $\phi^{\text{eff}}(t)$ into $\pi\pi$, which is given by (1.8) with (1.7) and (1.5), we take for the instantaneous transition rate of the prepared state $\phi^+(t)$ into $\pi\pi$ the

matrix element $\langle \pi\pi | H_w | \phi^+_{y=1}(t) \rangle$, where H_w is the CP-conserving interaction Hamiltonian. The difference between the standard phenomenology and the model that we want to consider in this section is the following: Our Hamiltonian is

$$H = H_0 + H_w \qquad \text{with} \qquad [H, CP] = 0, \qquad [H_w, CP] = 0 \quad (4.1)$$

and our K^0 state [$\phi^-_{y=1}$ prepared in the inelastic scattering process (3.1)] is given by (3.27), which evolves with a preferred time direction, $t \geq 0$. In the effective theory summarized in Section 1, the Hamiltonian is H given by (1.2), but the time-evolving state vector is given by (1.6) [and though it is only used for the forward time direction, there is no theoretical reason that $\phi^{\text{eff}}(t)$ could not also evolve backward in time]. For our model we define the ratio $R(t)$ by replacing in (1.8) H_{int} by H_w, ϕ^{eff} by ϕ^+, and $\langle \pi\pi | \mathrm{H}_{\text{int}} | K_S \rangle$ by $\langle \pi\pi | H_w | K_1^- \rangle$, since $|K_S^-\rangle \to |K_1^-\rangle$ for an H given by (4.1). Thus the normalized instantaneous rate as a function of t (proper time in the K^0 rest frame) is given by[11]

$$R(t) = \left| \frac{\langle \pi\pi | H_w | \phi^+_{y=+1}(t) \rangle}{\langle \pi\pi | H_w | K_1^- \rangle} \right|^2 \quad (4.2)$$

We calculate $R(t)$ using our theoretical result (3.27) and obtain [by inserting (3.27) into (4.2)]

$$R(t) = \left| e^{(-\gamma_1/2)t} e^{-im_1 t} \frac{b}{\sqrt{2}} + \int_0^{-\infty \mathrm{II}} dE \, \frac{\langle \pi\pi | H_w | E, cp = +1^- \rangle}{\langle \pi\pi | H_w | K_1^- \rangle} \frac{b(E)}{\sqrt{2}} e^{-iEt} \right|^2 \quad (4.3)$$

since $\langle \pi\pi | H_w | K_2^- \rangle = 0$, and $\langle \pi\pi | H_w | E, cp = -1, {}^- \rangle = 0$ due to (4.1). In the same way as for the $R(t)$ of (1.9), the first term vanishes for large values of $t \approx 20\ (1/\gamma_s)$, $(e^{-\gamma_s t/2} \approx 10^{-5})$, so that

$$R(t = 20\tau_s) = \left| \frac{1}{\sqrt{2}\langle \pi\pi | H_w | K_1^- \rangle} \times \int_0^{-\infty \mathrm{II}} dE \, \langle \pi\pi | H_w | E, cp = +1^- \rangle b(E) e^{-iEt} \right|^2 \quad (4.4)$$

[11] To be more scrupulous, we should calculate the time-dependent transition rate $\mathcal{P}(t)$ using the exact Golden Rule of RHS quantum mechanics (Bohm and Gadella, 1989; Bohm, 1994), which is possible and will be done in a future publication. But in order not to complicate the presentation further by novel arguments which are not relevant for the problem under consideration, we use this semiheuristic $R(t)$ and the observable parameters η_{+-} (and η_{00}) defined in terms of the effective theory for the phenomenological analysis of our exact (untruncated) theory (3.27).

Theoretically, not much can be said at this stage about the integral on the r.h.s. of (4). We write it using (3.22) as

$$I(t) = -\int_{-\infty_{II}}^{0} dE\, \langle \pi\pi | H_w | E, cp = +1^- \rangle\, S_{cp=+1}(E) \langle {}^+E, cp = +1 | \phi^+ \rangle e^{-iEt}$$

$$= -\int_{-\infty_{II}}^{0} dE\, \langle \pi\pi | H_w | E, cp = +1^+ \rangle \langle {}^+E, cp = +1 | \phi^+ \rangle e^{-iEt} \tag{4.5}$$

Then (4.3) can be written as

$$R(t) = \left| e^{-(\gamma_1/2)t} e^{-im_1 t} \frac{b}{\sqrt{2}} + \frac{I(t)}{\langle \pi\pi | H_w | K_1^- \rangle \sqrt{2}} \right|^2 \tag{4.6}$$

The integral $I(t)$ describes that part of the transition $\phi^+ \to \pi\pi$ which does not go through K_1^- resonance formation. We know that $\langle {}^+E | \phi^+ \rangle \in \mathcal{S} \cap \mathcal{H}_-$ (Hardy class function of the lower half-plane, second sheet). If we also knew that $\langle \pi\pi | H_w | E^- \rangle S(E) \in \mathcal{S} \cap \mathcal{H}_-$, then we could prove (using the Riemann–Lebesgue lemma for Hardy class functions) that $I(t)$ also decreases for increasing t, but it decreases less than an exponential $e^{-\gamma t}$. Even though we do not have this information, let us assume that the nonresonant background $I(t)$ will survive the exponential $e^{-\gamma_1 t}$.

$$|I(t)|^2 \geq \text{const} \cdot e^{-\gamma_1 t} \qquad \text{for} \qquad t \geq 20 \frac{1}{\gamma_1} \tag{4.7}$$

The magnitude of $\langle {}^+E_{II}, cp = 1 | \phi^+ \rangle$ [which can be calculated from $\langle {}^+E | \phi^+ \rangle$ on the positive real axis, i.e., at physical values of E, because of its Hardy class property (Bohm and Gadella, 1989; Bohm, 1994)] depends upon ϕ^+, i.e., upon the preparation of the K^0 state. Since experimentally the $\pi^+\pi^-$ are selected such that their invariant mass is near the center of a K^0 Breit–Wigner energy distribution, the magnitude of the nonresonance contributions $|\langle {}^+E_{II} | \phi^+ \rangle|$ to the observed $R(t)$ will be small. But we need only a small contribution on the r.h.s. of (4.4) in order to explain the experimental value (1.13) for $R(20\tau_s)$. Mathematically, $\langle {}^+E_{II} | \phi^+ \rangle$ must be different from zero, because $\phi^+ \in \Phi_-$. But its magnitude could be arbitrarily small, too small to account even for the small value (1.13) of $R(20\tau_1)$. Thus the question is not whether such a term $|I(t)/\langle \pi\pi(H_w | K_1^- \rangle|$, which decreases more slowly in time than the exponential $e^{-\gamma_1 t}$, exists, but whether this term in (4.6) has the right magnitude to explain the value (1.13).

Since we have no theoretical prediction, we shall use the second term on the r.h.s. of (4.6) as phenomenological parameter.

The phenomenological value that we obtain for this background integral from the values (1.12)[12] and (1.13) is

$$\left|\frac{I(t = 20\tau_1)}{\langle\pi\pi|H_w|K_1^-\rangle}\right| \approx 2.23 \times 10^{-3} \tag{4.8}$$

This means that if the nonresonant contributions to the transitions of the prepared state ϕ^+ into $\pi\pi$ are about 2.23×10^{-3} of the K_1-resonance term, then these contributions can explain the Princeton effect without the assumption of a *CP*-violating Hamiltonian. Theoretically $|I(t)/\langle\pi\pi|H_w|K_1^-\rangle|$ could of course be much smaller than 10^{-3}; in fact it can be arbitrarily small as long as it is not equal to zero and still fulfill the mathematical conditions that $\phi^+ \in \Phi_-$.

To display the distinction between the way the effective Lee–Oehme–Yang theory explains the existence of a long-time $K^0 \to \pi\pi$ decay mode and the way the exact theory in the RHS explains this effect, we compare the state vector for K-short and K-long in both theories. According to (1.1), we have (except for some normalization)

$$|K_S^-\rangle = |K_1\rangle + \epsilon_S|K_2\rangle \tag{4.9a}$$

$$|K_L^-\rangle = \epsilon_L|K_1\rangle + |K_2\rangle \tag{4.9b}$$

Here $|K_S^-\rangle$ and $|K_L^-\rangle$, and not $|K_1\rangle$ and $|K_2\rangle$, are assumed to be the Gamow vectors, and $\epsilon_L \approx \epsilon_S \approx \epsilon \approx 10^{-3}$ are very small. In analogy to this we define

$$\phi_\sigma^+ = |K_1^-\rangle + \frac{1}{b} F_{cp=-1}^- \tag{4.10a}$$

$$\phi_\lambda^+ = \frac{1}{b} F_{cp=1}^- + |K_2^-\rangle \tag{4.10b}$$

where $|K_{1,2}^-\rangle$ are the Gamow vectors and F_{cp}^- are the vectors (3.17) in the expansion (3.16). Then ϕ_y^+ of (3.24), (3.25) can be written as

$$\phi_{y=\pm 1}^+ = \phi_\sigma^+ \frac{b}{\sqrt{2}} \pm \phi_\lambda^+ \frac{b}{\sqrt{2}} \tag{4.11}$$

This formula is the analogue of (1.5) (with $a_S = \pm a_L = b/\sqrt{2}$) of the effective theory.

[12]The observable parameters η_{+-} (and η_{00}) are by (1.10) defined in terms of the effective theory. But they are experimentally determined as the ratio of the $K^0 \to \pi^+\pi^-$ rate at long times $t \approx 20(1/\gamma_s)$ to the rate at short times ($t = 1/\gamma_s$ extrapolated back to $t = 0$), when the $K^0 \to \pi^+\pi^-$ are mainly due to $K_S \to \pi^+\pi^-$. Thus the value (1.12) for η_{+-} is also in the present case related by (1.11) with sufficient accuracy to $R(t = 20\tau_S)$.

Comparing (4.10) with (4.11), we see that:

- ϕ_σ^+ are, like the $|K_S\rangle$, mostly $cp = +1$ vectors with a small $cp = -1$ contribution.
- ϕ_λ^+ are, like the $|K_L\rangle$, mostly $cp = -1$ vectors with a small $cp = +1$ contribution.

But whereas in (4.9) the small admixtures of the opposite parity is given by $\epsilon|K_2\rangle \in \mathcal{H}_2$ and $\epsilon|K_1\rangle \in \mathcal{H}_2$, respectively, the opposite parity admixtures in (10), $(1/b)F^-_{cp=-1}$ and $(1/b)F^-_{cp=+1}$, respectively, are not elements of $\mathcal{H}_2$. (Recall that $\mathcal{H}_2 \subset \Phi^\times_+$ is the space spanned by the vectors $|K_1^-\rangle$, $|K_2^-\rangle$). Since in the effective theory one has only the space $\mathcal{H}_2$, one had to postulate an effective Hamiltonian $\mathbb{H}$ with $[\mathbb{H}, CP] \neq 0$ in order to obtain (4.9b) and

$$\langle\pi\pi|\mathbb{H}|K_L^-\rangle = \epsilon\langle\pi\pi|\mathbb{H}|K_1^-\rangle + \langle\pi\pi|\mathbb{H}|K_2^-\rangle \neq 0 \tag{4.12}$$

In the infinite-dimensional RHS formulation one has other vectors outside of $\mathcal{H}_2$ which have the opposite parity, so that

$$\langle\pi\pi|H\phi_\lambda^+\rangle = \frac{1}{b}\langle\pi\pi|H|F^-_{cp=+1}\rangle \neq 0 \tag{4.13}$$

even though $[H, CP] = 0$. This term gives the $\pi\pi$ transitions for large $t \approx 20\gamma_s$ when the amplitude $\langle\pi\pi|H|\phi_\sigma^+\rangle$ is suppressed by a factor of $e^{-\gamma_1 t/2}$. Thus the analogue of $\epsilon\langle\pi\pi|\mathbb{H}|K_1^-\rangle$ is in the RHS formulation given by $(1/b)\langle\pi\pi|H|F^-_{cp=+1}\rangle$, which has nothing to do with CP violation. In the exact RHS theory with non-CP-violating H, however, there cannot be an anologue of the direct CP-violation amplitude $\langle\pi\pi|\mathbb{H}|K_2^-\rangle$.

The theoretical CP problems that the Princeton effect caused for the Lee–Oehme–Yang theory clearly had its origin in the confinement of the theory to the two-dimensional space $\mathcal{H}_2$. This confinement to $\mathcal{H}_2$ is an ad hoc assumption which cannot be justified in an infinite-dimensional space of states, except as the zeroth order of perturbation theory: If one considers $H_0 = H - H_w$ as the exact Hamiltonian and H_w as a perturbation of H and if one takes the eigenvectors $|K_1^-\rangle$ and $|K_2^-\rangle$ of H as the zeroth-order eigenvectors of H_0, then one knows from perturbation theory that the higher order eigenvectors of H_0 are generally not linear combinations of the zeroth-order eigenvectors. The exact eigenvectors $|K^0\rangle$, $|\bar{K}^0\rangle$ of H_0 are then given by the highest (∞) order of perturbation theory. They therefore cannot in general be linear combinations of the zeroth eigenvectors $|K_1^-\rangle$ and $|K_2^-\rangle$ only. But this is exactly what would be required in (1.5) (for $a_S = a_L = 1/\sqrt{2}$ or $\phi^{\text{eff}} = |K^0\rangle$). Though the complex basis vector expansion (B.2) was the origin of the idea that the Princeton effect can be explained without a CP-violating Hamiltonian, it is not really needed in order to ask the question why the prepared state $|K^0\rangle$ should be expandable (with an accuracy of 10^{-3}) only in

terms of vectors of the two-dimensional space $\mathcal{H}_2$. The existence of a finite (>2) or infinite number of linearly independent (basis) vectors—irrespective of what definition of convergence one uses—is already sufficient to see the problem. The RHS is needed to explain the existence of the eigenvectors (1.3a) with the property (1.4a) for $t \geq 0$ and to justify the inclusion of these eigenvectors in a complete basis system for the prepared state vectors $\phi^+_{y=1} \in \Phi_-$ (where $\phi^+_{y=1}$ corresponds to $|K^0\rangle$). For these properties one needs the mathematical (topological) completion of the linear scalar product space Ψ (Appendix A).

5. THE STANDARD PHENOMENOLOGICAL DESCRIPTION OF *CP* VIOLATION AS A TRUNCATION OF THE EXACT THEORY IN THE RHS

Even if the $\pi\pi$ decay of the long-lived neutral K-meson state can be explained without a CP-violating interaction Hamiltonian, it does not mean that there is no CP violation of the Hamiltonian. As was already mentioned above, the background term in (3.24)—though it needs to be there—may be so small $[|\sqrt{\gamma_1}b(E)/b| << 10^{-3}]$ that it cannot account for the Princeton effect. Also, the observed time dependence of the transition amplitude (1.8), in particular the interference term in (1.9), may be such that it cannot be explained by the background integrals in (4.3) or (3.27). Therefore, we want to apply now the same exact theory of Sections 2 and 3 to the CP-violating Hamiltonian

$$\mathrm{H} = H + H_{sw} + H_0 + H_w + H_{sw}, \qquad [\mathrm{H}, CP] \neq 0$$

Then, in place of the eigenvectors $|K_1^-\rangle$ and $|K_2^-\rangle$ of H, we use the eigenvectors $|K_S^-\rangle$ and $|K_L^-\rangle$ of H in the complex basis vector expansion (B.2) of the K-meson state vector ϕ^+_y:

$$\phi^+ = \frac{1}{\sqrt{2}}(|K_s^-\rangle b_s + |K_L^-\rangle b_L) + \sum_\beta \int_0^{-\infty \mathrm{II}} dE\, |E, \beta^-\rangle\, b_\beta(E)/\sqrt{2} \tag{5.1}$$

Here $|E, \beta^-\rangle$ are the generalized eigenvectors of H, and β are the degeneracy quantum numbers (where now $\beta \neq cp$). The complex basis vector expansion (5.1) is, as before, very general and exact, under the assumption that $|K_S^-\rangle$ and $|K_L^-\rangle$ are the only Gamow vectors with the right quantum numbers for the neutral K-system. The complex expansion coefficients b_S, b_L, and $b_\beta(E) = \langle E, \beta|\phi^+\rangle$ depend again predominantly upon the experimental conditions

for the preparation of ϕ^+. Again for mathematical reasons $b_\beta(E) \in \mathscr{H}_-^2 \cap \mathscr{S}$ cannot be exactly zero, but it could be arbitrarily small (mathematics provides no information about the order of magnitude involved). Thus, if $|\sqrt{\gamma_i}b(E)/b_{S,L}| << 10^{-3}$, then, as far as the Princeton effect (which is of order 10^{-3}) is concerned, every pure neutral-K state obtained from (5.1) is adequately approximated as

$$\phi^+ \approx (|K_S^-\rangle b_S + |K_L^-\rangle b_L)\frac{1}{\sqrt{2}} \sim |K_S^-\rangle\rho + |K_L^-\rangle \tag{5.2}$$

Since the time evolution of the Gamow vectors $|K_S^-\rangle$ and $|K_L^-\rangle$ is derived (not assumed) to be given by (1.4b) (with the additional result that $t \geq 0$ and with the qualification (3.7)), we have obtained in the RHS an exact theory which contains the standard phenomenological description of the neutral K-system with CP violation as an approximation. No new physics has been developed, but the standard phenomenological description has been given an exact theoretical foundation. The background terms in (5.1) may play (an observable) role in other experimental investigations (Maini, 1992; LeGac, 1996; Huet and Pesken, 1995; Ellis *et al.*, 1984, 1992, 1995).

Since the basis vectors on the r.h.s. of (5.1) are generalized eigenvectors of H, the time evolution operator $e^{-i\mathrm{H}^\times t}$ is diagonal:

$$\phi^+(t) = e^{-i\mathrm{H}^\times t}\phi^+ = \frac{1}{\sqrt{2}}(e^{-im_S t}e^{-(\gamma_S/2)t}|K_S^-\rangle b_S + e^{-im_L t}e^{-(\gamma_L/2)t}|K_L^-\rangle b_L)$$
$$+ \sum_\beta \int_0^{-\infty_{\mathrm{II}}} dE\, e^{-iEt}|E, \beta^-\rangle b^\beta(E)/\sqrt{2}; \qquad t \geq 0 \tag{5.3}$$

In particular, $|K_L^-\rangle$ cannot evolve by its own Hamiltonian H (i.e., without additional interaction with a regenerator) into $|K_S^-\rangle$ or vice versa (i.e., there is no "vacuum regeneration of K_S from K_L"), and neither can K_S be regenerated due to $e^{-i\mathrm{H}t}$ from the background term

$$F_\beta^- = \int_0^{-\infty_{\mathrm{II}}} dE\, E, \beta^-\rangle b^\beta(E)/\sqrt{2}$$

The Gamow vectors $|K_S^-\rangle$ and $|K_L^-\rangle$ evolve (as a consequence of their definition from the resonance poles) irreversibly and obey the exact exponential decay law (1.4b):

$$e^{-i\mathrm{H}^\times t}|K_S^-\rangle = e^{-im_S t}e^{-(\gamma_S/2)t}|K_S^-\rangle, \qquad t \geq 0$$
$$e^{-i\mathrm{H}^\times t}|K_L^-\rangle = e^{-im_L t}e^{-(\gamma_L/2)t}|K_L^-\rangle, \qquad t \geq 0 \tag{5.4}$$

There is no additional term on the r.h.s. of (5.4), in contrast to exact infinite-dimensional theories in the Hilbert space (Khalfin, 1972, 1994). Also,

$|K_S^-\rangle$ and $|K_L^-\rangle$ cannot be expressed in terms of $\phi^+_{y=+1}$ and $\phi^+_{y=-1}$, or any other finite or infinite superposition of $\phi_y^+ \in \Phi_-$, due to the terms $F_\beta^- \in \Phi_+^\times$ on the r.h.s. of (5.1). The time evolution in (5.3) is irreversible, $t \geq 0$, and $\phi^+(t) \in \Phi_+^\times$. This means that it can only be evaluated as a functional $\langle \psi^- | \phi^+(t) \rangle$ at $\psi^- \in \Phi_+$ (which represents observed decay products like $\psi^- = |\pi\pi\rangle$). In particular, the functional $\phi^+(t)$ cannot be evaluated at $\phi^+ \in \Phi_-$; i.e., the quantity $\langle \phi^+_{y'} | \phi^+_y(t) \rangle = \langle \phi^+_{y'} | e^{-iH^\times t} | \phi^+_y \rangle$, which would represent a vacuum regeneration amplitude of $K^0(y' = -1)$ from $K^0(y = +1)$, and vice versa, makes no sense in our theory. These kinds of quantities also have no observable meaning, since no experiment can measure the probability for a "transition" from an in-state $\phi^+_{y=1}$ into another in-state $\phi^+_{y=-1}$.

6. SUMMARY AND CONCLUSIONS

The purpose of this paper was to use the neutral K-system of two interfering resonances to test some aspects of the RHS quantum theory of microphysical irreversibility. We limited our investigation to the hypothesis that the two decaying K-states are ordinary, first-order, S-matrix-pole resonances [by the choice of (B.2)] since the standard theory with complex effective Hamiltonian makes the same hypothesis (by the choice of a diagonalizable complex Hamiltonian matrix rather than a Jordan block). Then we saw that the effective Lee–Oehne–Yang theory is a subtheory of the exact theory in the RHS. It must be emphasized that this is not the case for the exact theory in Hilbert space, because the Hilbert space theory does not allow for a complex basis vector expansion. As a bonus, we saw that the remainder of the exact theory, which is always ignored in the two-dimensional effective theory, leads to a nonzero 2π decay rate of the neutral K-system for large time even if we choose a CP-conserving Hamiltonian. This may or may not be of practical significance, since at this stage nothing can be said about its magnitude.

Many more experimental properties are known about the instantaneous transition rate $|\langle \pi\pi | H_{\text{int}} | \phi^+(t) \rangle|^2$ than have been used in our discussion in this paper (Gibbons, 1993; Geweniger *et al.*, 1974; Christenson *et al.*, 1974; Barr *et al.*, 1993; Gibbons *et al.*, 1993). To make adequate use of these properties, the background terms $\langle \pi\pi | H_w | F_{cp}^-(t) \rangle$ need to be investigated further and more of its characteristics need to be known than just the property that it decreases slower than exponentially in time. Of particular interest is the transition rate at instances around $t = 12\tau_s$, where it has been fitted (see above references) to the interference term $\cos(\Delta m t + \varphi)$ of (1.9), a result which, in that form, probably cannot be obtained from the background term. These questions will have to be discussed in a subsequent paper.

APPENDIX A. FROM A PRE-HILBERT SPACE TO A RIGGED HILBERT SPACE

A pre-Hilbert space is a linear space Ψ with a scalar product. This scalar product is denoted by

$$(\psi, F) \qquad \text{or by} \qquad (\psi|F) \tag{A.1}$$

The pre-Hilbert space is without any topological structure; that means neighborhoods, the convergence of infinite sequences, topological completeness, continuous operators, continuous functionals, dense subspaces, etc., are not defined. This space is what physicists mostly use for their calculations (together with a few additional rules) when they speak of the Hilbert space.

The Hilbert space $\mathcal{H}$ of mathematicians is a much more complicated structure. In order to make it topologically complete, its elements are not represented by functions (wave functions), but by classes of functions whose elements differ on a set of Lebesgue measure zero, a mathematically complicated and physically useless concept (because the apparatus resolution is described by a smooth function, not a set of Lebesque square-integrable functions). The RHS is the same linear space Ψ, only with different topological completions: one completes Ψ with respect to a topology that is stronger than the topology given by the Hilbert-space norm (e.g., one uses a countable number of norms) to obtain the space $\Phi \subset \mathcal{H}$ and considers in addition the topological dual to Φ, i.e., the space of *continuous* antilinear functionals of Φ denoted by $\Phi^\times$. Then one obtains the triplet of completions of Ψ (all differing from Ψ only by limit elements), the Gelfand triplet, or rigged Hilbert space:

$$\Phi \subset \mathcal{H} = \mathcal{H}^\times \subset \Phi^\times \tag{A.2}$$

$$\begin{aligned} &\text{with elements "bra" and "ket"} \quad \langle\phi| \in \Phi, \quad |F\rangle \in \Phi^\times \\ &\text{or "ket" and "bra"} \quad |\phi\rangle \in \Phi, \quad \langle F| \in \Phi^\times \end{aligned} \tag{A.3}$$

One widespread example for Φ is the Schwartz space $\mathcal{S}$ (i.e., Φ is often "realized" by the space of functions $\mathcal{S}$).

The vectors $\phi \in \Phi$ (in their form as either kets $|\phi\rangle$ or bras $\langle\phi|$) represent physical quantities connected with the experimental apparatuses [e.g., a state ϕ defined by a preparation apparatus or an observable $|\psi\rangle\langle\psi|$ defined by a registration apparatus (detector) satisfies $\phi, \psi \in \Phi$], the vectors $\langle F|$ or $|F\rangle \in \Phi^\times$ represent quantities connected with the microphysical system (e.g. "scattering states" $|E\rangle$ or decaying states $|E - i\Gamma/2\rangle$). But there are vectors in $\mathcal{H}$, namely those whose wave functions are not smooth, which cannot be related to experimental quantities.

A general observable is now represented by a bounded operator A in Φ (but in general by an unbounded operator $\overline{A}$ or $A^\dagger$ in $\mathcal{H}$) and corresponding to the triplet (A.2) one has now a triplet of operators

$$A^\dagger|_\Phi \subset A^\dagger \subset A^\times \tag{A.4}$$

Here $A^\dagger$ is the Hilbert space adjoint of A (if A is essentially self-adjoining, then $A^\dagger = \overline{A}$), $A^\dagger|_\Phi$ is its restriction to the space Φ, and the operator $A^\times$ in $\Phi^\times$ is the conjugate operator of A defined by

$$\langle A\phi|F\rangle = \langle \phi|A^\times F\rangle \qquad \text{for all} \quad \phi \in \Phi \quad \text{and all} \quad |F\rangle \in \Phi^\times \tag{A.5}$$

By this definition, $A^\times$ is the extension of the operator $A^\dagger$ to the space $\Phi^\times$ (and not the extension of the operator A which is most often used in mathematics). A very important point is that the operator $A^\times$ is only defined for an operator A which is continuous (and bounded) in Φ; then $A^\times$ is a continuous (but not bounded) operator in $\Phi^\times$. It is impossible in quantum mechanics (empirically) to restrict oneself to continuous (and therefore bounded) operators $\overline{A}$ in $\mathcal{H}$. But one can restrict oneself to algebras of observables $\{A, B, \ldots\}$, described by continuous operators in Φ, if the topology of Φ is suitably chosen. Then $A^\times$, $B^\times$, ... are defined and continuous in $\Phi^\times$. If A in (A.5) is not self-adjoint, then $A^\dagger|_\Phi$ need not be a continuous operator in Φ even if A is, but one can still define the conjugate $A^\times$ which is continuous in $\Phi^\times$.

A generalized eigenvector $F \in \Phi^\times$ of an operator A is defined by

$$\langle A\phi|F\rangle = \langle \phi|A^\times F\rangle = \omega\langle \phi|F\rangle \qquad \text{for all} \quad \phi \in \Phi \tag{A.6}$$

where the complex number ω is called the generalized eigenvalue. This is also written as

$$A^\times|F\rangle = \omega|F\rangle \tag{A.7}$$

For an essentially self-adjoint operator ($A^\dagger = \overline{A}$ = closure of A) this is often also written (following Dirac) as

$$A|F\rangle = \omega|F\rangle \tag{A.8}$$

especially if one suppresses the mathematical subtleties and acts as if one has just a linear scalar-product space Ψ. The generalized eigenvalues ω for self-adjoint operators $A^\dagger$ need not be real.

APPENDIX B. THE COMPLEX BASIS VECTOR EXPANSION

The most important result of the new mathematical theory of quantum physics in the rigged Hilbert space is the complex eigenvector expansion.

This is the generalization of the elementary basis vector expansion of a 3-dimensional vector,

$$\mathbf{x} = \sum_{i=1,2,3} \mathbf{e}_i(\mathbf{e}_i \cdot \mathbf{x}) = \sum \mathbf{e}_i x_i$$

to the expansion of vectors $\phi^+ \in \Phi_-$ using as basis vectors the generalized eigenvectors $|z_{R_i}^-\rangle$ of self--adjoint operators H with complex eigenvalues z_{R_i} and z.

Earlier developments toward this generalization include the fundamental theorem of linear algebra, which states that for every self-adjoint operator H in an n-dimensional Euclidean space $\mathcal{H}_n$ there exists an orthonormal basis $e_i \cdots e_n$ in $\mathcal{H}_n$ of eigenvectors $He_i = E_i e_i$, i.e., $f \in \mathcal{H}_n$ can be written as $f = \Sigma_{i=1}^n e_i(e_i, f)$. This theorem generalizes to the infinite-dimensional Hilbert space $\mathcal{H}$, but only for self-adjoint operators H which are completely continuous (also called compact operators which include Hilbert–Schmidt, nuclear, trace-class operators). For an arbitary self-adjoint operator one has to go outside the space to find a complete basis system of eigenvectors (which are then called generalized).

The *first* step in this direction is the Dirac basis vector expansion, which in mathematical terms is called the nuclear spectral theorem. It states that for every $\phi \in \Phi$

$$\phi = \int_0^{+\infty} dE\, |E^+\rangle\langle {}^+E|\phi^+\rangle + \sum_n |E_n)(E_n|\phi) \qquad \text{for} \quad \phi \in \Phi \qquad \text{(B.1)}$$

Here, $|E_n)$ are the discrete eigenvectors of the exact Hamiltonian $H = K + V$ (describing the bound states), $H|E_n) = E_n|E_n)$, and $|E^+\rangle$ are the generalized eigenvectors (Dirac kets) of H fulfilling $\langle H\chi|E^+\rangle = \langle \chi|H^\times|E^+\rangle = E\langle\chi|E^+\rangle$ for all $\chi \in \Phi$; cf. (A.6). The "coordinates" of the vector ϕ with respect to the continuous basis $|E^+\rangle$, i.e., the set of energy wave functions $\langle {}^+E|\phi^+\rangle$, form a "realization" of the space Φ by a space of functions. We call $\phi \in \Phi$ "well-behaved" if $\langle {}^+E|\phi^+\rangle$ is a well-behaved function, i.e., of the Schwartz space $\mathcal{S}$. The $|E^+\rangle$ correspond to the continuous spectrum (describing scattering states) and the integration extends over the spectrum of H: $0 \leq E < \infty$. In place of the $|E^+\rangle$, one could also have chosen the $|E^-\rangle$ if the *out*-wavefunctions are more readily available.

The *second* step is the "complex basis vector expansion." It holds for "very well-behaved" vectors of a subspace Φ_- of Φ (Schwartz space). For every $\phi^+ \in \Phi_-$ (a similar expansion holds also for every $\psi^- \in \Phi_+$) one obtains for the case of a finite number of resonance poles of the analytically

continued S-matrix at the positions $z_{R_i} = E_{R_i} - i\Gamma_i/2$, $i = 1, 2, \ldots, N$, the following basis system expansion:

$$\phi^+ = \int_0^{-\infty_{II}} d\omega \; |\omega^+\rangle\langle^+\omega|\phi^+\rangle + \sum_{i=1}^{N} |z_{R_i}^-\rangle 2\pi\Gamma_i \langle^+ z_{R_i}|\phi^+\rangle$$
$$+ \sum_n |E_n)(E_n|\phi) \qquad \text{for} \quad \phi^+ \in \Phi_- \tag{B.2}$$

where $|z_{R_i}^-\rangle\sqrt{2\pi\Gamma_i} = \psi^{G_i} \in \Phi^\times_+$ are Gamow kets (C.1) representing decaying states (C.2). Their properties are summarized in Appendix C below (see also Bohm and Gadella, 1989; Bohm, 1994). The forms (B.1) and (B.2) of the generalized basis vector expansions assume that H is the only observable to be diagnalized (cyclic operator). If the complete system of commuting observables (c.s.c.o.) consists of $H, B_1, B_2, \ldots, B_N \equiv H, B$, then the projection operators $|E_n)$ $(E_n| \to \Sigma\, |E_n, b)$ $(E_n, b|$, where the sum extends over all values of the degeneracy quantum numbers b of the energy E_n. Similarly in (B.1), (B.2)

$$|E^+\rangle\langle^+E| \to \sum_b |E, b^+\rangle\langle^+E, b|$$
$$|z_R^-\rangle\langle^+ z_R| \to \sum_b |z_R^-, b\rangle\langle^+ z_R, b|$$

The operator B could be, e.g., the hypercharge operator *if* $[H, B] = 0$; it can be the operator CP if $[H, CP] = 0$.

We will from now on omit the last sum in (B.1) and (B.2), as it represents the sum over the stationary, bound states, which have no importance for the problem of this paper (no bound states appear). Then we have two exact but different basis vector expansions for the same ϕ^+ [if we choose $\phi = \phi^+ \in \Phi_- \subset \Phi$ in (B.1)]: (B.1) is the standard expansion and has a correspondence in the Hilbert space (spectral resolution of operators with a continuous spectrum), while (B.2) is new and shows that the quasistationary states $|z_{R_i}^-\rangle$ can serve as basis vectors in very much the same manner as the stationary states $|E_n)$ in the standard case. But in addition to the resonance states, the new basis vector expansion (B.2) also contains an integral over the negative real axis from, e.g., $E = 0$ to $-\infty_{II}$ in the second sheet of the energy surface of the S-matrix. This integral, called the "background term" ϕ_{bg}^+ [which may be as much a misnomer as the term "complex spectral resolution" for (B.2)], may be infinitesimally small, but cannot be zero. But it may also have some small but observable consequences. It can be calculated using the van Winter theorem (Bohm and Gadella, 1989; Bohm, 1994) from the values $\langle^+E|\phi^+\rangle$ for physical energies $0 \leq E < \infty$ and depends upon the apparatus for ϕ^+ and ψ^-.

APPENDIX C. GAMOW VECTORS

Gamow vectors are generalizations of Dirac kets, and therefore we denote them also by kets $|\psi^G\rangle = |z_R^-\rangle\sqrt{2\pi\Gamma}$, where $z_R = E - i\Gamma/2$ is the complex energy value (for every $|z_R^-\rangle \in \Phi_+^\times$ there is also a Gamow vector $|z_R^{*+}\rangle \in \Phi_-^\times$, $z_R = E - i\Gamma/2$). The Gamow vectors have the following properties:

1. They are generalized eigenvectors of Hamiltonians H [which we always assume to be (essentially) self-adjoint and bounded from below] with generalized eigenvalues $z_R = E_R - i\Gamma/2$,

$$H^\times|\psi^G\rangle = z_R|\psi^G\rangle \tag{C.1}$$

   where E_R and Γ are interpreted, respectively, as the resonance energy and width.

2. They satisfy the following exponential decay law for $t \geq 0$ only:

$$\begin{aligned} W^G(t) &= e^{-iH^\times t}|\psi^G\rangle\langle\psi^G|e^{iHt} \\ &= e^{-i(E_R - i\Gamma/2)t}|\psi^G\rangle\langle\psi^G|e^{i(E_R + i\Gamma/2)t} = e^{-\Gamma t}W^G(0) \end{aligned} \tag{C.2}$$

3. They have a Breit–Wigner energy distribution.
4. They obey an exact Golden Rule of which Fermi's Golden Rule is the Born approximation.
5. They are associated with a pole at z_R in the second sheet of the analytically continued S-matrix. They are derived as the functionals of the pole term of the S-matrix.

In the absence of a vector description of resonances in the Hilbert space formulation, the pole of the S-matrix has commonly been taken as the definition of a resonance. In the RHS formulation the Gamow vectors are derived from the pole term of the S-matrix (Bohm, 1978, 1981, 1994; Gadella, 1983a,b, 1984; Antoniou, 1992; Antoniou and Prigogine, 1993; Antoniou and Tasaki, 1993; Bohm and Gadella, 1994), and therefore these vectors $|z_R^-\rangle \in \Phi_+^\times$ describe decaying resonances as autonomous microphysical entities, in very much the same way as the $|E_n\rangle$ describe stable particles. (There are also Gamow vectors $|z_R^{*+}\rangle$, $z_R^* = E_R + i\Gamma/2$, associated with the pole at z_R^*, and they have an exponentially growing semigroup evolution for $-\infty < t \leq 0$.)

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REFERENCES

Antoine, J. P. (1969a). *Journal of Mathematical Physics*, **10**, 53.
Antoine, J. P. (1969a). *Journal of Mathematical Physics*, **10**, 2276.
Antoniou, I. (1992). In *Proceedings of the II International Wigner Symposium, Goslar, 1991*, H. D. Doebner *et al.*, eds., World Scientific, Singapore.
Antoniou, I., and Prigogine, I. (1993). *Physica A*, **192**, 443.
Antoniou, I., and Tasaki, S. (1993). *International Journal of Quantum Chemistry*, **46**, 425.
Barr, G. D., *et al.* (1993). *Physics Letters B*, **317**, 233.
Bohm, A. (1967). *Boulder Lectures in Theoretical Physics 1966*, Vol. 9A, Gordon and Breach, New York.
Bohm, A. (1978). *Letters in Mathematical Physics*, **3**, 455.
Bohm, A. (1981). *Journal of Mathematical Physics*, **22**, 2813.
Bohm, A. (1993). In *Proceedings of the Symposium on the Foundations of Modern Physics, Cologne, June 1, 1993*, P. Busch and P. Mittelstaedt, eds., World Scientific, Singapore, p. 77.
Bohm, A. (1994). *Quantum Mechanics*, 3rd ed., Springer, New York.
Bohm, A. (1995). *Physical Review A*, **51**, 1758.
Bohm, A., and Gadella, M. (1989). *Dirac Kets, Gamov Vectors and Gel'fand Triplets*, Springer, Berlin.
Bohm, A., Darriulat, P., Grasso, C., Kaftanov, V., Kleinknecht, K., Lynch, H. L., Rubbia, C., Ticho, H., and Tittel, K. (1969). *Nuclear Physics B*, **9**, 605.
Bohm, A., Antoniou, I., and Kielanowski, P. (1994). *Physics Letters A*, **189**, 442.
Bohm, A., Antoniou, I., and Kielanowski, P. (1995). *Journal of Mathematical Physics*, **36**, 2593.
Bohm, A., Maxson, S., Loewe, M., and Gadella, M. (1996). Quantum mechanical irreversibility, *Physica A*, to appear.
Christenson, J., Cronin, J., Fitch, V., and Turley, R. (1964). *Physical Review Letters*, **13**, 138.
Christenson, J. H., *et al.* (1974). *Physical Review Letters*, **43**, 1212.
Ellis, J., Hagelin, J. S., Nanopoulos, D. V., and Sredicki, M. (1984). *Nuclear Physics B*, **241**, 381.
Ellis, J., Lopez, J. L., Mavromatos, N. E., and Nanopolous, D. V. (1992). *Physics Letters B*, **293**, 142.
Ellis, J., Lopez, J. L., Mavromatos, N. E., and Nanopolous, D. V. (1995). CERN-TH 95–99.
Fonda, L., Ghirardi. G. C., and Rimini, A. (1978). *Reports on Progress in Physics*, **41**, 587.
Gadella, M. (1983a). *Journal of Mathematical Physics*, **24**, 2142.
Gadella, M. (1983b). *Journal of Mathematical Physics*, **24**, 1462.
Gadella, M. (1984). *Journal of Mathematical Physics*, **25**, 2481.
Gamow, G. (1928). Zeitschrift für Physik, **51**, 510.
Garcia-Calderon, G., and Peierls, R. (1976). *Nuclear Physics A*, **265**, 443.
Geweniger, C., *et al.* (1974). *Physics Letters*, **48B**, 487.
Gibbons, L. K. (1993). Dissertation, University of Chicago, August 1993.
Gibbons, L. K., *et al.* (1993). *Physical Review Letters*, **70**, 1199, 1203.

Hernandez, E., and Mondragon, A. (1984). *Physical Review C*, **29**, 722.
Huet, P., and Peskin, M. E. (1995). *Nuclear Physics B*, **434**, 3.
Khalfin, L. A. (1972). *JETP Letters*, **25**, 349.
Khalfin, L. A. (1994). CP violation problem beyond the Lee–Oehme–Yang theory, in *The DaΦne Physics Handbook*, Frascati, Italy.
Kleinknecht, K. (1989). In *CP Violation*, C. Jarskog, ed., World Scientific, Singapore, p. 41.
Lee, T. D. (1981). *Particle Physics and Introduction to Field Theory*, Harwood, Chur, Switzerland, and references therein.
LeGac, R. (1996). Study of *CP*, *T* and *CPT* in the neutral *K* system, in Workshop on *K* Physics, Orsay, May 1996.
Ludwig, G. (1983/1985). *Foundations of Quantum Mechanics*, Vols. I and II, Springer-Verlag, Berlin.
Maiani, L., Pancheri, G., and Paver, N. (1992). *The DaΦne Physics Handbook*, Vol. 1, Servizio documentazione dei Laboratori nazionali di Frascati, Frascati, Italy.
Melsheimer, O. (1974). *Journal of Mathematical Physics*, **15**, 902.
Pais, A. (1990). In *CP-Violation: The First 25 Years*, J. Tran Thanh Van, ed., Editions Frontières, Gif-sur-Yvette, France.
Peierls, R. E. (1955). In *Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics*, E. M. Bellamy *et al.*, eds., Pergamon Press, Oxford.
Roberts, E. (1966). *Journal of Mathematical Physics*, **7**, 1097.